

# SPECTRAL PAIRS IN CARTESIAN COORDINATES

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*Dedicated to the memory of Irving E. Segal*

ABSTRACT. Let  $\Omega \subset \mathbb{R}^d$  have finite positive Lebesgue measure, and let  $\mathcal{L}^2(\Omega)$  be the corresponding Hilbert space of  $\mathcal{L}^2$ -functions on  $\Omega$ . We shall consider the exponential functions  $e_\lambda$  on  $\Omega$  given by  $e_\lambda(x) = e^{i2\pi\lambda \cdot x}$ . If these functions form an orthogonal basis for  $\mathcal{L}^2(\Omega)$ , when  $\lambda$  ranges over some subset  $\Lambda$  in  $\mathbb{R}^d$ , then we say that  $(\Omega, \Lambda)$  is a *spectral pair*, and that  $\Lambda$  is a *spectrum*. We conjecture that  $(\Omega, \Lambda)$  is a spectral pair if and only if the translates of some set  $\Omega'$  by the vectors of  $\Lambda$  tile  $\mathbb{R}^d$ . In the special case of  $\Omega = I^d$ , the  $d$ -dimensional unit cube, we prove this conjecture, with  $\Omega' = I^d$ , for  $d \leq 3$ , describing all the tilings by  $I^d$ , and for all  $d$  when  $\Lambda$  is a discrete periodic set. In an appendix we generalize the notion of spectral pair to measures on a locally compact abelian group and its dual.

## 1. INTRODUCTION

The setting of *spectral pairs* in  $d$  real dimensions involves two subsets  $\Omega$  and  $\Lambda$  in  $\mathbb{R}^d$  such that  $\Omega$  has finite and positive  $d$ -dimensional Lebesgue measure, and  $\Lambda$  is an index set for an orthogonal  $\mathcal{L}^2(\Omega)$ -basis  $e_\lambda$  of exponentials, i.e.,

$$(1.1) \quad e_\lambda(x) = e^{i2\pi\lambda \cdot x}, \quad x \in \Omega, \lambda \in \Lambda$$

where  $\lambda \cdot x = \sum_{j=1}^d \lambda_j x_j$ . We use vector notation  $x = (x_1, \dots, x_d)$ ,  $\lambda = (\lambda_1, \dots, \lambda_d)$ ,  $x_j, \lambda_j \in \mathbb{R}$ ,  $j = 1, \dots, d$ . The basis property refers to the Hilbert space  $\mathcal{L}^2(\Omega)$  with inner product

$$(1.2) \quad \langle f | g \rangle_\Omega := \int_\Omega \overline{f(x)} g(x) \, dx$$

where  $dx = dx_1 \cdots dx_d$ , and  $f, g \in \mathcal{L}^2(\Omega)$ . The corresponding norm is

$$(1.3) \quad \|f\|_\Omega^2 := \langle f | f \rangle_\Omega = \int_\Omega |f(x)|^2 \, dx,$$

as usual. It follows that the spectral pair property for a pair  $(\Omega, \Lambda)$  is equivalent to the nonzero elements of the set

$$\Lambda - \Lambda = \{\lambda - \lambda' : \lambda, \lambda' \in \Lambda\}$$

being contained in the *zero-set* of the complex valued function

$$(1.4) \quad z \longmapsto \int_\Omega e^{i2\pi z \cdot x} \, dx =: F_\Omega(z)$$

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and the corresponding  $e_\lambda$ -set  $\{e_\lambda : \lambda \in \Lambda\}$  being *total* in  $\mathcal{L}^2(\Omega)$ . Recall, totality means that the span of the  $e_\lambda$ 's is dense in  $\mathcal{L}^2(\Omega)$  relative to the  $\|\cdot\|_\Omega$ -norm, or, equivalently, that  $f = 0$  is the only  $\mathcal{L}^2(\Omega)$ -solution to:

$$\langle f | e_\lambda \rangle_\Omega = 0, \quad \text{for all } \lambda \in \Lambda.$$

Note,  $F_\Omega(z)$  is defined for any  $z = (z_1, \dots, z_d) \in \mathbb{R}^d$  since  $\Omega$  has finite measure and  $e^{i2\pi z \cdot x}$  has absolute value  $= 1$ . We refer to the book [Ped97] for a summary of the theory of *spectral pairs*. It was developed in the previous joint papers by the coauthors [JoPe87, JoPe91, JoPe92, JoPe93a, JoPe93b, JoPe94, JoPe95, JoPe96] and elsewhere, e.g., [LRW98, LaWa96, LaWa97a, LaWa97b]. We recall that Fuglede showed [Fug74] that the disk and the triangle in two dimensions are *not spectral sets*, in the sense that, if  $\Omega$  is one of these sets, then there is *no* possible choice for  $\Lambda$  such that  $(\Omega, \Lambda)$  is a spectral pair in  $\mathbb{R}^2$ .

If  $\Omega \subset \mathbb{R}^d$  is open, then we consider the partial derivatives  $\frac{\partial}{\partial x_j}$ ,  $j = 1, \dots, d$ , defined on  $C_c^\infty(\Omega)$  as unbounded skew-symmetric operators in  $\mathcal{L}^2(\Omega)$ . The corresponding versions  $\frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$  are symmetric of course. We say that  $\Omega$  has the *extension property* if there are commuting self-adjoint extension operators  $H_j$ , i.e.,

$$(1.5) \quad \frac{1}{i} \frac{\partial}{\partial x_j} \subset H_j, \quad j = 1, \dots, d.$$

We have (see [Fug74, Jor82, Ped87, JoPe92])

**Theorem 1.1.** (Fuglede, Jorgensen, Pedersen) *Let  $\Omega \subset \mathbb{R}^d$  be open and connected with finite and positive Lebesgue measure. Then  $\Omega$  has the extension property if and only if it is a spectral set. If  $\Omega$  is only assumed open, then the spectral-set property implies the extension property, but not conversely.*

Some of the interest in spectral pairs derives from their connection to *tilings*. A subset  $\Omega \subset \mathbb{R}^d$  with nonzero measure is said to be a *tile* if there is a set  $L \subset \mathbb{R}^d$  such that the translates  $\{\Omega + l : l \in L\}$  cover  $\mathbb{R}^d$  up to measure zero, and if the intersections

$$(1.6) \quad (\Omega + l) \cap (\Omega + l') \quad \text{for } l \neq l' \text{ in } L$$

have measure zero. We will call  $(\Omega, L)$  a *tiling pair* and we will say that  $L$  is a *tiling set*. The *Spectral-Set conjecture* due to Fuglede (see [Fug74, Jor82, Ped87, JoPe95, LaWa96, LaWa97a, LaWa97b]) states:

**Conjecture 1.2.** Let  $\Omega \subset \mathbb{R}^d$  have positive and finite Lebesgue measure. Then  $\Omega$  is a spectral set if and only if  $\Omega$  is a tile, i.e., there exists a set  $L$  so that  $(\Omega, L)$  is a spectral pair if and only if there exists a set  $L'$  so that  $(\Omega, L')$  is a tiling pair.

We formulate a “dual” conjecture.

**Conjecture 1.3.** Let  $L \subset \mathbb{R}^d$ . Then  $L$  is a spectrum if and only if  $L$  is a tiling set, i.e., there exists a set  $\Omega$  so that  $(\Omega, L)$  is a spectral pair if and only if there exists a set  $\Omega'$  so that  $(\Omega', L)$  is a tiling pair.

**Conjecture 1.4.** Let  $L \subset \mathbb{R}^d$ . Then  $(I^d, L)$  is a spectral pair if and only if  $(I^d, L)$  is a tiling pair.

The significance of the special case  $\Omega = I^d$  lies in part in the results below where we show, for  $d = 1, 2, 3$ , that  $(I^d, \Lambda)$ ,  $\Lambda \subset \mathbb{R}^d$ , is a *spectral pair* if and only if  $I^d$

tiles  $\mathbb{R}^d$  by  $\Lambda$ -translates. Our proofs also construct all possible spectra for the unit cube when  $d = 1, 2, 3$ . In Section 4 we establish Conjecture 1.4 for all  $d$  when  $\Lambda$  is a discrete periodic set.

Tiling questions for  $I \subset \mathbb{R}$  are trivial, but not so for  $I^d \subset \mathbb{R}^d$  when  $d \geq 2$ . The connection between *tiles* and *spectrum* is more direct for  $\Omega = I^d$  than for other examples of sets  $\Omega$ . This is explained by the following (easy) lemma relating the problems to the function  $F_\Omega$  from (1.4) above.

**Lemma 1.5.** *If  $\Omega = I^d$ , then the zero-set for the function  $F_\Omega$  in (1.4) is*

$$(1.7) \quad \mathbf{Z}_{I^d} = \{z \in \mathbb{C}^d \setminus \{0\} : \exists j \in \{1, \dots, d\} \text{ s.t. } z_j \in \mathbb{Z} \setminus \{0\}\}.$$

*Proof.* The function  $F_{I^d}(\cdot)$  factors as follows.

$$(1.8) \quad F_{I^d}(z) = \prod_{j=1}^d \frac{e^{i2\pi z_j} - 1}{i2\pi z_j}$$

for  $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ , with the interpretation that the function  $z \mapsto \frac{e^{i2\pi z} - 1}{i2\pi z}$  is 1 when  $z = 0$  in  $\mathbb{C}$ .  $\square$

In particular, if  $(I^d, \Lambda)$  is a spectral pair, then  $\Lambda - \Lambda \subset \mathbf{Z}_{I^d} \cup \{0\}$ . The corresponding result for tilings is non-trivial, it was proved by Keller [Kel30, Kel37], a detailed proof appears in [Per40]. The precise statement of Keller's theorem is:

**Theorem 1.6.** *If  $(I^d, \Lambda)$  is a tiling pair, then  $\Lambda - \Lambda \subset \mathbf{Z}_{I^d} \cup \{0\}$ , where  $\mathbf{Z}_{I^d}$  is given by (1.7).*

Let  $\mu, \nu$  be two Borel measures on  $\mathbb{R}^d$ . We will say that  $(\mu, \nu)$  is a *tiling pair* if the convolution,  $\mu * \nu$ , of  $\mu$  and  $\nu$  is Lebesgue measure on  $\mathbb{R}^d$ . This coincides with the previous definition of a tiling pair in the sense that if  $(\Omega, L)$  is a pair of subsets of  $\mathbb{R}^d$  so that  $\Omega$  has finite positive Lebesgue measure,  $L$  is discrete,  $\omega$  denotes Lebesgue measure restricted to  $\Omega$ , and  $\ell$  denotes counting measure on  $L$ , then  $(\Omega, L)$  is tiling pair if and only if  $(\omega, \ell)$  is a tiling pair. Since convolution is commutative,  $(\mu, \nu)$  is a tiling pair if and only if  $(\nu, \mu)$  is a tiling pair. In the appendix we introduce (and investigate properties of) a notion of a spectral pair of measures  $(\mu, \nu)$ . In particular, we show that  $(\mu, \nu)$  is a spectral pair if and only if  $(\nu, \mu)$  is a spectral pair.

After this paper was originally submitted two independent proofs [LRW98], [IoPe98] of Conjecture 1.4 have appeared.

## 2. CONSTRUCTION OF SPECTRA

The next two sections are concerned with the structure of the discrete sets  $\Lambda$  which at the same time serve as spectra for  $I^d$  (i.e., the basis property), and also are sets of vectors  $\lambda$  which make the translates  $\lambda + I^d$  tile  $\mathbb{R}^d$ .

There is a *recursive procedure* for constructing spectral pairs in higher dimensions from “*factors*” in lower dimension. It is a *cross-product* construction, and it applies to any two spectral pairs,  $(\Omega_i, \Lambda_i)$ ,  $i = 1, 2$ , in arbitrary dimensions  $d_1$  and  $d_2$ . While it is clear that the “spectral-pair category” is *closed under tensor product* (see [JoPe92, JoPe94]), the following result is new:

**Theorem 2.1.** *Let  $(\Omega_1, \Lambda_1)$  be a spectral pair in dimension  $d_1$ , let  $\Omega_2$  be a set of finite positive measure in dimension  $d_2$ . Suppose that for each  $\lambda_1 \in \Lambda_1$ ,  $\Lambda(\lambda_1)$  is*

a discrete subset of  $\mathbb{R}^{d_2}$  such that  $(\Omega_2, \Lambda(\lambda_1))$  is a spectral pair. If  $\Lambda = \{(\lambda_1, \lambda_2) : \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda(\lambda_1)\}$  then  $(\Omega_1 \times \Omega_2, \Lambda)$  is a spectral pair in  $d_1 + d_2$  dimensions.

*Proof.* We first show that the exponentials  $\{e_\lambda : \lambda \in \Lambda\}$  are mutually orthogonal in  $\mathcal{L}^2(\Omega_1 \times \Omega_2)$  where the  $e_\lambda$ 's are given on  $\Omega_1 \times \Omega_2$  by the usual formula (1.1) from Section 1. The inner product in  $\mathcal{L}^2(\Omega_1 \times \Omega_2)$  of  $e_\lambda$  and  $e_{\lambda'}$  factors as follows:

$$\int_{\Omega_1} e_{\lambda_1 - \lambda'_1}(x) \left( \int_{\Omega_2} e_{\lambda_2 - \lambda'_2}(y) dy \right) dx.$$

If  $\lambda_1 \neq \lambda'_1$  in  $\Lambda_1$ , then it vanishes since  $(\Omega_1, \Lambda_1)$  is a spectral pair; and, if  $\lambda_1 = \lambda'_1$  but  $\lambda_2 \neq \lambda'_2$ , it vanishes since  $(\Omega_2, \Lambda(\lambda_1))$  is one. This proves orthogonality of  $\Lambda$ . To see that it is total, let  $f \in \mathcal{L}^2(\Omega_1 \times \Omega_2)$  and suppose  $f$  is orthogonal to  $\Lambda$ . The inner products (vanishing) are:

$$\langle e_\lambda | f \rangle_{\Omega_1 \times \Omega_2} = \int_{\Omega_2} e_{\lambda_2}(y) e_{\lambda_1}(y) \left( \int_{\Omega_1} e_{\lambda_1}(x) \overline{f(x, y)} dx \right) dy.$$

If  $\lambda_1$  is fixed, and the double integral vanishes for all  $\lambda_2 \in \Lambda(\lambda_1)$ , then the integral  $\int_{\Omega_1} e_{\lambda_1}(x) \overline{f(x, y)} dx = 0$  for almost all  $y$ , by the totality of  $\Lambda(\lambda_1)$  on  $\Omega_2$ . But  $\lambda_1$  is arbitrary so the totality of  $\Lambda_1$  on  $\Omega_1$  implies  $f = 0$ . We conclude, that  $\Lambda$  is total on  $\Omega_1 \times \Omega_2$  as claimed.  $\square$

A more concrete version of Theorem 2.1 is:

**Theorem 2.2.** *Let  $(\Omega_i, \Lambda_i)$ ,  $i = 1, 2$ , be spectral pairs in the respective dimensions  $d_1$  and  $d_2$ , and let  $\beta : \Lambda_1 \rightarrow \mathbb{R}^{d_2}$  be an arbitrary function. Let*

$$(2.1) \quad \Lambda_\beta := \left\{ \begin{pmatrix} \lambda_1 \\ \beta(\lambda_1) + \lambda_2 \end{pmatrix} : \lambda_1 \in \Lambda_1 \text{ and } \lambda_2 \in \Lambda_2 \right\}.$$

*Then  $(\Omega_1 \times \Omega_2, \Lambda_\beta)$  is a spectral pair in  $d_1 + d_2$  dimensions.*

*Proof.* If  $(\Omega_2, \Lambda_2)$  is a spectral pair, then so is  $(\Omega_2, \Lambda_2 + \beta)$  for any vector  $\beta$ . An application of Theorem 2.1 completes the proof.  $\square$

By repeatedly applying Theorem 2.2 it follows that if  $\Lambda$  is the set of points given by:

$$(2.2) \quad \begin{pmatrix} \alpha + k_1 \\ \beta_1(k_1) + k_2 \\ \beta_2(k_1, k_2) + k_3 \\ \vdots \\ \beta_{d-1}(k_1, k_2, \dots, k_{d-1}) + k_d \end{pmatrix}$$

with  $k_1, k_2, \dots, k_d \in \mathbb{Z}$ , where  $\beta_i : \mathbb{Z}^i \rightarrow [0, 1)$  are fixed functions, then  $(I^d, \Lambda)$  is a spectral pair. Of course, there are the obvious modifications resulting from permutation of the  $d$  coordinates; but, when  $d \geq 10$ , these configurations do *not* suffice for cataloguing all the possible spectra  $\Lambda$  which turn  $(I^d, \Lambda)$  into an  $\mathbb{R}^d$ -spectral pair; see Section 4.

We now turn to a result which is a partial converse to Theorem 2.1, its statement requires the following notation. It is motivated by the “projection method” from

quasicrystal theory; see, e.g., [Hof95]. For subsets  $\Lambda$  of the Cartesian product  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ , let

$$\Lambda_1 := P_1\Lambda = \left\{ \lambda_1 \in \mathbb{R}^{d_1} : \exists \lambda_2 \in \mathbb{R}^{d_2} \text{ s.t. } \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in \Lambda \right\}$$

(where it is convenient here to use column vector formalism). We shall also need sections of  $\Lambda$  in the  $\mathbb{R}^{d_2}$  coordinate direction as follows: If  $\lambda_1 \in \Lambda_1 (= P_1\Lambda)$ , set

$$\Lambda(\lambda_1) := \left\{ \lambda_2 \in \mathbb{R}^{d_2} : \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in \Lambda \right\}.$$

**Lemma 2.3.** *Let  $\Omega_i \subset \mathbb{R}^{d_i}$ ,  $i = 1, 2$ , be subsets with finite positive Lebesgue measure in the respective dimensions, and let  $\Lambda \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$  be a subset such that  $(\Omega_1 \times \Omega_2, \Lambda)$  is a spectral pair in  $d_1 + d_2$  dimensions. Then for every  $\lambda_1 \in P_1\Lambda$ , the exponentials  $\{e_\xi^{(2)} : \xi \in \Lambda(\lambda_1)\}$  are orthogonal in  $\mathcal{L}^2(\Omega_2)$ ; and they are total in  $\mathcal{L}^2(\Omega_2)$  if and only if  $e_{\lambda'_1}^{(1)}$  and  $e_{\lambda'_1}^{(1)}$  are orthogonal in  $\mathcal{L}^2(\Omega_1)$  for all  $\lambda'_1 \in P_1(\Lambda) \setminus \{\lambda_1\}$ .*

*Proof.* To check orthogonality, let  $\xi, \eta \in \Lambda(\lambda_1)$ . Then the two points  $\begin{pmatrix} \lambda_1 \\ \xi \end{pmatrix}$  and  $\begin{pmatrix} \lambda_1 \\ \eta \end{pmatrix}$  are in  $\Lambda$ , and the corresponding  $\Omega_1 \times \Omega_2$ -inner product is zero. But it is also

$$m_{d_1}(\Omega_1) \langle e_\xi | e_\eta \rangle_{\Omega_2},$$

and since  $m_{d_1}(\Omega_1) > 0$ , the orthogonality follows.

We now show that  $\Lambda(\lambda_1)$  is total in  $\mathcal{L}^2(\Omega_2)$  if  $\langle e_{\lambda'_1}^{(1)} | e_{\lambda'_1}^{(1)} \rangle_{\Omega_1} = 0$  for all  $\lambda'_1 \in P_1\Lambda$ ,  $\lambda'_1 \neq \lambda_1$ . Let  $g \in \mathcal{L}^2(\Omega_2)$  and suppose  $g$  is orthogonal to all the  $\Lambda(\lambda_1)$ -exponentials. Let  $\begin{pmatrix} \lambda'_1 \\ \lambda'_2 \end{pmatrix}$  be a general point in  $\Lambda$ . Then the inner product with  $e_{\lambda'_1}^{(1)} \otimes g$  is

$$\langle e_{\lambda'_1}^{(1)} | e_{\lambda'_1}^{(1)} \rangle_{\Omega_1} \langle e_{\lambda'_2}^{(2)} | g \rangle_{\Omega_2}.$$

If  $\lambda'_1 = \lambda_1$ , then  $\lambda'_2 \in \Lambda(\lambda_1)$ , and the second factor vanishes. If  $\lambda'_1 \neq \lambda_1$ , then the first factor is zero, and we get that  $e_{\lambda'_1}^{(1)} \otimes g$  is orthogonal in  $\mathcal{L}^2(\Omega_1 \times \Omega_2)$  to the  $\Lambda$ -exponentials. They are total, and we conclude that  $g$  vanishes in  $\mathcal{L}^2(\Omega_2)$ .

The remaining case is when  $\langle e_{\lambda'_1}^{(1)} | e_{\lambda'_1}^{(1)} \rangle_{\Omega_1} \neq 0$  for some  $\lambda'_1 \in P_1(\Lambda) \setminus \{\lambda_1\}$ . But it follows that then  $\langle e_\xi^{(2)} | e_\eta^{(2)} \rangle_{\Omega_2} = 0$  for all  $\xi \in \Lambda(\lambda_1)$  and  $\eta \in \Lambda(\lambda'_1)$ . In particular,  $\Lambda(\lambda_1)$  is then *not* total in  $\mathcal{L}^2(\Omega_2)$ .  $\square$

### 3. DIMENSIONS TWO AND THREE

In this section we prove Conjecture 1.4 for  $d = 1, 2, 3$ . Furthermore we give a complete classification of the possible spectra for the unit cube in those dimensions.

We begin with the following simple observation in one dimension for  $\Omega = I = [0, 1)$ .

**Proposition 3.1.** *The only subsets  $\Lambda \subset \mathbb{R}$  such that  $(I, \Lambda)$  is a spectral pair are the translates*

$$(3.1) \quad \Lambda_\alpha := \alpha + \mathbb{Z} = \{\alpha + n : n \in \mathbb{Z}\}$$

where  $\alpha$  is some fixed real number.

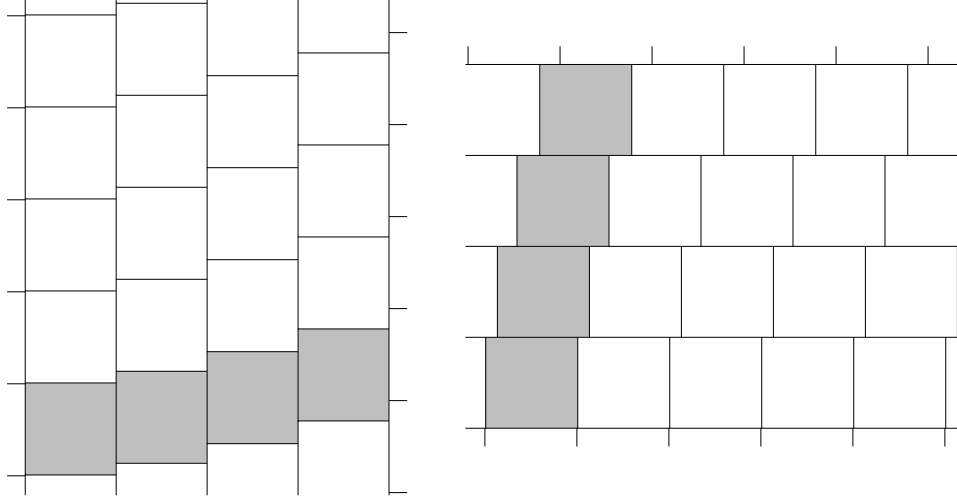


FIGURE 1. Illustrating tiling with (3.2) (left) and (3.3) (right)

In two dimensions, the corresponding result is more subtle, but the possibilities may still be enumerated as follows:

**Theorem 3.2.** *The only subsets  $\Lambda \subset \mathbb{R}^2$  such that  $(I^2, \Lambda)$  is a spectral pair must belong to either one or the other of the two classes, indexed by a number  $\alpha$ , and a sequence  $\{\beta_m \in [0, 1] : m \in \mathbb{Z}\}$ , where*

$$(3.2) \quad \Lambda = \left\{ \begin{pmatrix} \alpha + m \\ \beta_m + n \end{pmatrix} : m, n \in \mathbb{Z} \right\}$$

or

$$(3.3) \quad \Lambda = \left\{ \begin{pmatrix} \beta_n + m \\ \alpha + n \end{pmatrix} : m, n \in \mathbb{Z} \right\}.$$

*Each of the two types occurs as the spectrum of a pair for the cube  $I^2$ , and each of the sets  $\Lambda$  as specified is a tiling set for the cube  $I^2$ .*

*Proof.* The assertion in the theorem about  $\Lambda$ -translations tiling the plane with  $I^2$  is clear from (3.2)–(3.3), and it is illustrated graphically in Figure 1.

It is immediate from Theorem 2.2 that each one of the two formulas (3.2)–(3.3) for  $\Lambda$  make  $(I^2, \Lambda)$  a spectral pair, and the main result is that there are not others. We show this directly by an examination of the possibilities for  $\Lambda$  which are implied by the inclusion

$$(3.4) \quad \Lambda - \Lambda \subset \mathbf{Z}_{I^2} \cup \{0\}$$

where  $\mathbf{Z}_{I^2}$  is read off from Lemma 1.5 above. Again a translation of  $\Lambda$  by a single vector in the plane will reduce the analysis to the case when  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  is in  $\Lambda$ . Let  $\lambda = (\xi, \eta) \in \Lambda$  and suppose  $\lambda \notin \mathbb{Z}^2$ . Then either  $\xi$  or  $\eta$  is not an integer. Suppose  $\eta$  is not an integer. Then  $\xi$  is a nonzero integer. Let  $\lambda' = (\xi', \eta')$  be an arbitrary point in  $\Lambda$ . If  $\xi'$  is not an integer, then  $\eta'$  is a nonzero integer, so  $\lambda - \lambda'$  is not in  $\mathbf{Z}_{I^2}$ , contradicting (3.4). So  $\xi' \in \mathbb{Z}$  for any  $\lambda' = (\xi', \eta')$  in  $\Lambda$ . To verify  $\Lambda$  is a subset of a set given by (3.2) for  $\alpha = 0$ , we need only check that if  $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$  and  $\begin{pmatrix} \xi' \\ \eta' \end{pmatrix}$

are different points in  $\Lambda$ , with  $\xi = \xi' \in \mathbb{Z}$ , then  $\eta - \eta' \in \mathbb{Z}$ . But recall  $\begin{pmatrix} 0 \\ \eta - \eta' \end{pmatrix} \in \mathbf{Z}_{I^2}$ , so it follows from Lemma 1.5 that  $\eta - \eta' \in \mathbb{Z} \setminus \{0\}$ . Since an orthonormal basis cannot be a strict subset of another orthonormal basis for the same space it follows that  $\Lambda$  is given by (3.2).  $\square$

Replacing the appeal to Lemma 1.5 in this proof with an appeal to Theorem 1.6 it follows that any tiling set  $\Lambda$  for the cube  $I^2$  must be given by (3.2)–(3.3), we leave the details for the reader. The fact that this simple tiling pattern for the cube  $I^d$  in  $d$  dimensions is broken for  $d = 10$  follows from examples of Lagarias and Shor [LaSh92]. It is shown there that for each  $d \geq 10$  there exists a tiling of  $\mathbb{R}^d$  by translates of  $I^d$  such that no two tiles have a complete facet in common. These examples also demonstrate, see Section 4, that if  $d \geq 10$ , then the corresponding combinations (2.2) do not supply all possible spectra for  $I^d$ .

The following result shows that spectra for  $I^3$  and tilings of  $\mathbb{R}^3$  by  $I^3$  are the same by fully determining each. No complete description of such tilings or spectra is known for  $d > 3$ .

**Theorem 3.3.** *( $I^3, \Lambda$ ) is a tiling pair, or a spectral pair, if and only if, after a possible translation by a single vector and a possible permutation of the coordinates  $(x_1, x_2, x_3)$ ,  $\Lambda$  can be brought into the following form: there is a partition of  $\mathbb{Z}$  into disjoint subsets  $A, B$  (one possibly empty) with associated functions*

$$\begin{aligned} \alpha_0 &: A \longrightarrow [0, 1], \\ \alpha_1 &: A \times \mathbb{Z} \longrightarrow [0, 1], \\ \beta_0 &: B \longrightarrow [0, 1], \\ \beta_1 &: B \times \mathbb{Z} \longrightarrow [0, 1] \end{aligned}$$

such that  $\Lambda$  is the (disjoint) union of

$$(3.5) \quad \begin{pmatrix} a \\ \alpha_0(a) + k \\ \alpha_1(a, k) + l \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b \\ \beta_1(b, n) + m \\ \beta_0(b) + n \end{pmatrix}$$

as  $a \in A, b \in B$ , and  $k, l, m, n \in \mathbb{Z}$ .

*Proof.* Suppose  $\Lambda$  is a tiling set for  $I^3$ . By the tiling property there exist functions  $\alpha, \beta, \gamma: \mathbb{Z}^3 \mapsto \langle -1, 0 \rangle$  so that

$$\Lambda = \left\{ \begin{pmatrix} n + \alpha(l, m, n) \\ m + \beta(l, m, n) \\ l + \gamma(l, m, n) \end{pmatrix} : l, m, n \in \mathbb{Z} \right\}.$$

Fix  $l, m \in \mathbb{Z}$  then Theorem 1.6 (Keller's theorem) implies  $\alpha(l, m, n)$  is independent of  $n$ , we will write  $\alpha(l, m)$  in place of  $\alpha(l, m, n)$  to indicate this independence. Similarly,  $\beta(l, m, n) = \beta(l, n)$  and  $\gamma(l, m, n) = \gamma(m, n)$ . Considering, for fixed  $n \in \mathbb{Z}$ , the intersection of the plane  $x_1 = n$  by cubes  $I^3 + \lambda, \lambda \in \Lambda$  it follows that the set

$$\Lambda_{1,n} = \left\{ \begin{pmatrix} m + \beta(l, n) \\ l + \gamma(m, n) \end{pmatrix} : l, m \in \mathbb{Z} \right\}$$

is a tiling set for  $I^2$  in  $\mathbb{R}^2$ . Hence, by our  $d = 2$  result (Theorem 3.2), either

$$\Lambda_{1,n} = \left\{ \begin{pmatrix} m + \tilde{\beta}(l, n) \\ l + \tilde{\gamma}(n) \end{pmatrix} : l, m \in \mathbb{Z} \right\} \quad \text{or} \quad \Lambda_{1,n} = \left\{ \begin{pmatrix} m + \tilde{\beta}(n) \\ l + \tilde{\gamma}(m, n) \end{pmatrix} : l, m \in \mathbb{Z} \right\}.$$

It follows that there exist  $A, B \subset \mathbb{Z}$  so that  $A \cup B = \mathbb{Z}$ ,  $A \cap B = \emptyset$ , and

$$\Lambda = \left\{ \begin{pmatrix} n + \alpha(l, m) \\ m + \beta(l, n) \\ l + \gamma(n) \end{pmatrix} : l, m \in \mathbb{Z}, n \in A \right\} \cup \left\{ \begin{pmatrix} n + \alpha(l, m) \\ m + \beta(n) \\ l + \gamma(m, n) \end{pmatrix} : l, m \in \mathbb{Z}, n \in B \right\}.$$

For each  $m \in \mathbb{Z}$

$$\Lambda_{2,m} = \left\{ \begin{pmatrix} n + \alpha(l, m) \\ l + \gamma(n) \end{pmatrix} : l \in \mathbb{Z}, n \in A \right\} \cup \left\{ \begin{pmatrix} n + \alpha(l, m) \\ l + \gamma(m, n) \end{pmatrix} : l \in \mathbb{Z}, n \in B \right\}$$

is a tiling set for  $I^2$  in  $\mathbb{R}^2$ . Hence, our  $d = 2$  result implies either (1)  $\alpha(l, m) = \alpha(m)$  for all  $l, m \in \mathbb{Z}$  or (2)  $\gamma(m, n) = \gamma(n)$ , for all  $m \in \mathbb{Z}$ ,  $n \in B$ . Suppose (1), then for each  $l \in \mathbb{Z}$

$$\Lambda_{3,l} = \left\{ \begin{pmatrix} n + \alpha(m) \\ m + \beta(l, n) \end{pmatrix} : m \in \mathbb{Z}, n \in A \right\} \cup \left\{ \begin{pmatrix} n + \alpha(m) \\ m + \beta(n) \end{pmatrix} : m \in \mathbb{Z}, n \in B \right\}$$

is a tiling set for  $I^2$ , hence our  $d = 2$  result implies that either (1a)  $\alpha(m) = \alpha_0$  for all  $m \in \mathbb{Z}$  or (1b)  $\beta(l, n) = \beta(l)$  for  $n \in A$  and  $\beta(n) = \beta_0$  for  $n \in B$ . If (1a) then we are done, so suppose (1b): then

$$\Lambda = \left\{ \begin{pmatrix} n + \alpha(m) \\ m + \beta(l) \\ l + \gamma(n) \end{pmatrix} : l, m \in \mathbb{Z}, n \in A \right\} \cup \left\{ \begin{pmatrix} n + \alpha(m) \\ m + \beta_0 \\ l + \gamma(m, n) \end{pmatrix} : l, m \in \mathbb{Z}, n \in B \right\}.$$

Let, if possible,  $n_1 \in A$ ,  $n_2 \in B$ ,  $m_1, m_2, l \in \mathbb{Z}$  be such that  $\alpha(m_1) \neq \alpha(m_2)$  and  $\beta(l) \neq \beta_0$ : then

$$\begin{pmatrix} n_1 + \alpha(m_1) \\ m_1 + \beta(l) \\ l + \gamma(n_1) \end{pmatrix} - \begin{pmatrix} n_2 + \alpha(m_2) \\ m_2 + \beta_0 \\ l + \gamma(n_2) \end{pmatrix}$$

does not have any nonzero integer entry, contradicting Keller's theorem. So, either  $B = \emptyset$ ,  $A = \emptyset$ ,  $\alpha(m) = \alpha_0$  for all  $m \in \mathbb{Z}$ , or  $\beta(l) = \beta_0$  for all  $l \in \mathbb{Z}$ ; in the three last cases we are done, so assume  $A = \mathbb{Z}$ . If  $\alpha(m_1) \neq \alpha(m_2)$ ,  $\beta(l_1) \neq \beta(l_2)$ , and  $\gamma(n_1) \neq \gamma(n_2)$  then

$$\begin{pmatrix} n_1 + \alpha(m_1) \\ m_1 + \beta(l_1) \\ l_1 + \gamma(n_1) \end{pmatrix} - \begin{pmatrix} n_2 + \alpha(m_2) \\ m_2 + \beta(l_2) \\ l_2 + \gamma(n_2) \end{pmatrix}$$

does not have any nonzero integer entry, contradicting Keller's theorem. This contradiction completes the proof of case (1). The proof of case (2) is similar; we leave the details for the reader.

Conversely, for every such set  $\Lambda$ , the translates of  $I^3$  by the vectors of  $\Lambda$  clearly tile  $\mathbb{R}^3$ . This completes the description of tilings of  $\mathbb{R}^3$  by  $I^3$ .

By Theorem 2.1 and Theorem 3.2 any set  $\Lambda$  of the form (3.5) is a spectrum for  $I^3$ . We sketch a proof of the converse.

Apply Lemma 2.3 to  $\Omega_1 \times \Omega_2 = I \times I^2$ , to show that if  $(I^3, \Lambda)$  is a spectral pair, then one of the three coordinate intervals may be picked as  $\Omega_1$  in Lemma 2.3, i.e.,  $\Omega_1 = I$ ,  $\Omega_2 = I^2 = I \times I$  and with  $P_1 \Lambda = \Lambda_1$  satisfying the orthogonality on  $\mathcal{L}^2(I)$ . By Lemma 1.5, this means

$$(3.6) \quad \Lambda_1 - \Lambda_1 \subset \mathbb{Z}.$$



Eventually we show that  $\Lambda_1$  must be of the form  $\theta_1 + \mathbb{Z}$ . But to select the one of the three coordinates which has this form, consider the canonical mapping

$$\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z} \simeq [0, 1)$$

and select the one of the three sets  $\pi(P_j(\Lambda))$ ,  $j = 1, 2, 3$ , of the smallest cardinality. Assume it is  $j = 1$  for simplicity. We will show that  $\Lambda_1$  then satisfies (3.6), so that Lemma 2.3 applies. The assertion is that the set  $\pi(\Lambda_1)$  is a singleton. The proof is indirect. Suppose *ad absurdum* that, for some  $\theta_1$  such that  $0 < \theta_1 < 1$ ,  $\Lambda_1$  meets both  $\mathbb{Z}$  and  $\theta_1 + \mathbb{Z}$ . We conclude from Theorem 3.2 that for  $\lambda_1$  in each of the two sets  $\mathbb{Z}$  or  $\theta_1 + \mathbb{Z}$ , the points in  $\Lambda(\lambda_1)$  must be of the form  $\binom{k}{\alpha(k)+l}$  for  $k, l \in \mathbb{Z}$  and  $\alpha: \mathbb{Z} \rightarrow [0, 1)$  some function, or alternatively  $\binom{\beta(l')+k'}{l'}$ ,  $k', l' \in \mathbb{Z}$  with  $\beta$  some possibly different function. Calculating  $\mathcal{L}^2(I^3)$ -inner products for associated points  $\lambda, \lambda' \in \Lambda$  with  $P_1(\lambda) = m \in \mathbb{Z}$ , and  $P_1(\lambda') = n + \theta_1$  ( $n \in \mathbb{Z}$ ), we get the following possibilities for the respective coordinates in the second and third place:

$$\binom{\alpha(m)+k}{\beta(m,k)+l} \quad \text{or} \quad \binom{\delta(m,l')+k'}{\gamma(m)+l'}.$$

But if the difference  $P_1\lambda - P_1\lambda'$  is not in  $\mathbb{Z}$ , then one of the corresponding differences in the second place, or the third place, must be in  $\mathbb{Z}$ . Making variations, we conclude that then one of the two sets  $\pi(P_j\Lambda)$ ,  $j = 2, 3$ , must be a singleton. But this contradicts that  $\pi(P_1\Lambda)$  has *two* distinct points, and is chosen to have smallest cardinality of the three sets  $\pi(P_j\Lambda)$ ,  $j = 1, 2, 3$ .  $\square$

**Corollary 3.4.** *The commuting self-adjoint extensions  $\{H_j : j = 1, \dots, d\}$  in (1.5) are completely classified and determined, for  $d = 1, 2, 3$  and  $\Omega = I^d$ , by Proposition 3.1 for  $d = 1$ , Theorem 3.2 for  $d = 2$ , and Theorem 3.3 for  $d = 3$ .*

*Proof.* The stated conclusion follows from combining the results in the present section with Theorem 1.1; for  $I^d$  the spectral condition is equivalent to the operator extension property.  $\square$

#### 4. PERIODIC SETS

A discrete set  $T \subset \mathbb{R}^d$  is *periodic* if there exists a finite set  $L \subset \mathbb{R}^d$  and an invertible  $d \times d$  matrix  $R$  with real entries such that  $T = L + R\mathbb{Z}^d$ . Periodic sets have played an important role in the study of spectral pairs, see, e.g., [JoPe92], [Ped96], [LaWa97b], and in the study of tilings by translation, see, e.g., [LaSh92], [LaWa96]. In this section we establish Conjecture 1.4 under the further hypothesis that  $L$  is a periodic set. The periodic case is of interest because ([Kel30]) Keller's conjecture about cube tilings (see below) is false if and only if it is false for certain periodic tilings. Also a long-standing conjecture is that any bounded tile in  $\mathbb{R}^d$  admits a periodic tiling set.

**Lemma 4.1.** *Let  $R$  be an invertible  $d \times d$  matrix with real entries and let  $\Lambda := R\mathbb{Z}^d$  be the corresponding lattice. If  $\Omega \subset \mathbb{R}^d$  is a measurable set, then  $\Omega$  is a  $\Lambda$ -tile if and only if*

- (i)  $m(\Omega) = |\det R|$  and
- (ii)  $m(\Omega \cap (\Omega + l)) = 0$  for all  $l \in \Lambda \setminus \{0\}$ ,

where  $m$  is the  $d$ -dimensional Lebesgue measure.

A measurable set  $\Omega$  is called a  $\Lambda$ -tile if  $\bigcup_{l \in \Lambda} (\Omega + l)$  is a measure-theoretic partition of  $\mathbb{R}^d$ . Note that  $\Omega_R := RI^d$  is a  $\Lambda$ -tile.

*Proof of Lemma 4.1.* If  $\Omega$  is a  $\Lambda$ -tile then  $m(\Omega) = m(\Omega_R) = |\det R|$ , since any two  $\Lambda$ -tiles have the same volume [GrLe87]. Conversely, suppose a measurable set  $\Omega$  has properties (i) and (ii). Let  $\Omega_l := (\Omega_R + l) \cap \Omega$ ; then  $\bigcup_{l \in \Lambda} \Omega_l$  is a measure-theoretic partition of  $\Omega$ . By (ii), the sets  $\Omega_l - l = \Omega_R \cap (\Omega - l)$ ,  $l \in \Lambda$ , are measure disjoint. Hence,

$$m\left(\bigcup_{l \in \Lambda} (\Omega_l - l)\right) = m\left(\bigcup_{l \in \Lambda} (\Omega_l)\right) = m(\Omega) = |\det R|$$

by (i). It follows that  $\bigcup_{l \in \Lambda} (\Omega_l - l) = \Omega_R$ , up to sets of measure zero. Hence, up to sets of measure zero,

$$\begin{aligned} \bigcup_{k \in \Lambda} (\Omega + k) &= \bigcup_{k \in \Lambda} \bigcup_{l \in \Lambda} (\Omega_l + k) = \bigcup_{l \in \Lambda} \bigcup_{k \in \Lambda} (\Omega_l + k) \\ &= \bigcup_{l \in \Lambda} \bigcup_{k \in \Lambda} (\Omega_l - l + k) = \bigcup_{k \in \Lambda} \bigcup_{l \in \Lambda} ((\Omega_l - l) + k) = \bigcup_{k \in \Lambda} (\Omega_R + k) = \mathbb{R}^d, \end{aligned}$$

as we needed to show.  $\square$

**Theorem 4.2.** *Let  $R$  be an invertible  $d \times d$  matrix with real entries, let  $\mathbf{L} \subset \mathbb{R}^d$  be finite with  $|\mathbf{L}|$  elements, and let  $L := R\mathbb{Z}^d$  be the lattice in  $\mathbb{R}^d$  generated by the columns of  $R$ . If  $L \cap (\mathbf{L} - \mathbf{L}) = \{0\}$ , then the following are equivalent:*

- (i)  $(I^d, L + \mathbf{L})$  is a spectral pair.
- (ii)  $I^d$  is an  $(L + \mathbf{L})$ -tile.
- (iii)  $|\mathbf{L}| = |\det R|$  and  $L + (\mathbf{L} - \mathbf{L}) \subset \mathbf{Z}_{I^d} \cup \{0\}$ .

*Proof.* (i)  $\Rightarrow$  (iii) Since the functions  $(e_\lambda)_{\lambda \in L + \mathbf{L}}$  are orthogonal it follows that  $L + (\mathbf{L} - \mathbf{L}) \subset \mathbf{Z}_{I^d} \cup \{0\}$ . By [Lan67, Theorem 4.2]  $L + \mathbf{L}$  has uniform density 1, hence  $|\mathbf{L}| = |\det R|$ . (iii)  $\Rightarrow$  (i) Using  $L + (\mathbf{L} - \mathbf{L}) \subset \mathbf{Z}_{I^d} \cup \{0\}$  we conclude that the functions  $(e_\lambda)_{\lambda \in L + \mathbf{L}}$  are orthogonal in  $\mathcal{L}^2(I^d)$ . So, since  $|\mathbf{L}| = |\det R|$ , an application of [Ped96, Theorem 1] allows us to conclude  $(I^d, L + \mathbf{L})$  is a spectral pair. (ii)  $\Rightarrow$  (iii) That  $L + (\mathbf{L} - \mathbf{L}) \subset \mathbf{Z}_{I^d} \cup \{0\}$  follows from  $I^d$  being an  $L + \mathbf{L}$ -tile, by Theorem 1.6. If  $l \neq l'$  in  $\mathbf{L}$ , then  $l - l'$  has a nonzero integer entry, hence  $m((I^d + l) \cap (I^d + l')) = 0$ . Hence  $I^d + \mathbf{L}$  has volume  $|\mathbf{L}|$ . By assumption  $I^d + \mathbf{L}$  is an  $L$ -tile, so using Lemma 4.1 we conclude  $|\mathbf{L}| = |\det R|$ . (iii)  $\Rightarrow$  (ii) Let  $x, x' \in L$ ,  $l, l' \in \mathbf{L}$ . If  $x \neq x'$  or  $l \neq l'$ , then  $(l - l') + (x - x') \in \mathbf{Z}_{I^d}$  so

$$(4.1) \quad m((I^d + x + l) \cap (I^d + x' + l')) = 0.$$

In particular  $I^d + \mathbf{L}$  has volume  $|\mathbf{L}| = |\det R|$  and  $m((I^d + \mathbf{L}) \cap (I^d + \mathbf{L} + x)) = 0$  for all  $x \in L \setminus \{0\}$ . Hence  $I^d + \mathbf{L}$  is an  $L$ -tile by Lemma 4.1. Using (4.1) it follows that  $I^d$  is an  $(L + \mathbf{L})$ -tile.  $\square$

While studying a conjecture of Minkowski, Keller [Kel30, Kel37] made the stronger conjecture that, if  $T \subset \mathbb{R}^d$  is any set such that  $I^d$  is a  $T$ -tile, then  $T - T$  contains one of the canonical unit basis vectors for  $\mathbb{R}^d$ . Keller's conjecture is true for  $d \leq 6$  [Per40] (see also [StSz94]), and false for  $d \geq 10$  [LaSh92]. It remains open for  $d = 7, 8, 9$ . The  $d = 10$  example in [LaSh92] has a set of translations of the form

$T = (2\mathbb{Z})^{10} + \mathbf{L}$ , where  $\mathbf{L}$  has  $2^{10} = 1024$  elements. To check (iii) in Theorem 4.2 it is sufficient to check that

$$\mathbf{L} - \mathbf{L} \subset \mathbf{Z}_{I^{10}} \cup \{0\},$$

but this is condition (a) in [LaSh92, p. 280], hence it is satisfied by construction. It follows that  $(I^{10}, T)$  is a spectral pair in  $\mathbb{R}^{10}$  such that  $T - T$  does not contain one of the canonical basis vectors; it follows that  $T$  is not of the form (2.2). In particular, we have verified that the sets  $\Lambda$  of the form (2.2) do not suffice for cataloguing all possible spectra for the unit cube  $I^{10}$  in  $\mathbb{R}^{10}$ .

#### APPENDIX: SPECTRAL PAIRS OF MEASURES

We extend the concept of a spectral pair to a spectral pair of measures  $(\mu, \nu)$ , where  $\mu$  is a Borel measure on a locally compact abelian group  $G$  and  $\nu$  a Borel measure on the dual group  $\Gamma$ . In the past, we have mainly studied spectral pairs in the situation  $G = \Gamma = \mathbb{R}^d$ ,  $\langle x, \xi \rangle = e^{i2\pi x\xi}$ ,  $\mu(X) = m(\Omega \cap X)$ , where  $m$  denotes Lebesgue measure, and  $\Omega \subset \mathbb{R}^d$  is Lebesgue measurable with  $m(\Omega) \neq 0$ . In studying this situation, we found it useful to also study spectral pairs in the cases  $G = \mathbb{Z}^d$ ,  $G = \mathbb{T}^d$ ,  $G = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ , and  $\mu$  a restriction of Haar measure on the respective groups  $G$ .

The present setup allows us to study all these cases simultaneously and to expose a fundamental symmetry between the “spectral set”  $\mu$  and the “spectrum”  $\nu$  in a spectral pair  $(\mu, \nu)$  of measures. We make the symmetric roles of  $\mu$  and  $\nu$  explicit, and we show that  $\mu(G) < \infty$  holds if and only if  $\nu$  has an atom.

Let  $G$  be a locally compact abelian group (written additively). Let  $m_G: G \rightarrow G$  be given by

$$(A.1) \quad m_G(x) := -x$$

for  $x \in G$ . If  $f: G \rightarrow \mathbb{C}$  is a complex-valued function defined on  $G$ , then we let

$$(A.2) \quad M_G f := f \circ m_G.$$

Let  $\mu$  be a positive Borel measure on  $G$ .

Let  $\tilde{\mu} := \mu \circ m_G^{-1}$ , i.e.,

$$(A.3) \quad \int f d\tilde{\mu} = \int M_G f d\mu.$$

Then  $M_G$  restricts to an isometric isomorphism of  $\mathcal{L}^2(\mu)$  onto  $\mathcal{L}^2(\tilde{\mu})$ .

Let  $\Gamma := \hat{G}$  be the dual group of the group  $G$ , i.e.,  $\Gamma$  is the set of all continuous homomorphisms of  $G$  into the unit circle  $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$ . Since  $\hat{\hat{G}} \simeq G$  [HeRo63], we can interpret  $G$  as a set of homomorphisms on  $\Gamma$ . We will write

$$(A.4) \quad \langle x, \xi \rangle \in \mathbb{T},$$

$x \in G$ ,  $\xi \in \Gamma$  for the duality between  $G$  and  $\Gamma$ . Then, for each  $\xi \in \Gamma$ ,  $e_\xi(x) := \langle x, \xi \rangle$  determines a continuous homomorphism  $G \rightarrow \mathbb{T}$ , and similarly  $e_x(\xi) := \langle x, \xi \rangle$  determines a continuous homomorphism  $\Gamma \rightarrow \mathbb{T}$ .

Define

$$(A.5) \quad (Ff)(\xi) := \int_G f(x) \overline{e_\xi(x)} d\mu(x)$$

for  $f \in \mathcal{L}^1 \cap \mathcal{L}^2(\mu)$ . Let  $\nu$  be a second positive Borel measure on  $\Gamma$ . If  $\{Ff : f \in \mathcal{L}^1 \cap \mathcal{L}^2(\mu)\}$  is a dense subset of  $\mathcal{L}^2(\nu)$  and

$$(A.6) \quad \int_{\Gamma} |Ff|^2 d\nu = \int_G |f|^2 d\mu$$

for each  $f \in \mathcal{L}^1 \cap \mathcal{L}^2(\mu)$ , then we say  $(\mu, \nu)$  is a *spectral pair (of measures)*. In the affirmative case,  $F (= F_{(\mu, \nu)})$  extends, by continuity, to an isometric isomorphism of  $\mathcal{L}^2(\mu)$  onto  $\mathcal{L}^2(\nu)$ .

Similarly to  $m_G$  and  $M_G$  in (A.1) and (A.2) above, we introduce  $m_{\Gamma}$  and  $M_{\Gamma}$ .

**Theorem A.1.** *If  $(\mu, \nu)$  is a spectral pair, then so is  $(\tilde{\nu}, \mu)$  with the transform  $\tilde{\nu}$  from  $\nu$  as introduced above.*

*Proof.* Let

$$(A.7) \quad (\tilde{F}g)(x) = \int_{\Gamma} g(\xi) \overline{e_x(\xi)} d\tilde{\nu}(\xi).$$

We must show that  $\tilde{F}$  extends, by continuity, to an isometric isomorphism, mapping  $\mathcal{L}^2(\tilde{\nu})$  onto  $\mathcal{L}^2(\mu)$ . If  $f \in \mathcal{L}^1 \cap \mathcal{L}^2(\mu)$  and  $g \in \mathcal{L}^1 \cap \mathcal{L}^2(\tilde{\nu})$ , then

$$(A.8) \quad \langle M_{\Gamma} Ff | g \rangle_{\tilde{\nu}} = \left\langle f \left| \tilde{F}g \right. \right\rangle_{\mu}$$

by a simple computation using the fact that  $M_{\Gamma}$  is an isometric isomorphism. Hence,  $\tilde{F}g = (M_{\Gamma} F)^* g$ . Since  $M_{\Gamma}$  and  $F$  both are isometric isomorphisms so is the adjoint  $(M_{\Gamma} F)^*$ .  $\square$

We have the following analogue of the usual Fourier inversion formula.

**Corollary A.2.** *If  $(\mu, \nu)$  is a spectral pair of measures and if  $g \in \mathcal{L}^1 \cap \mathcal{L}^2(\nu)$ , then*

$$(A.9) \quad (F^{-1}g)(x) = \int_{\Gamma} g(\xi) e_x(\xi) d\nu(\xi)$$

for  $\mu$ -a.e.  $x \in G$ .

*Proof.* In the proof of Theorem A.1, we showed that  $\tilde{F} = (M_{\Gamma} F)^*$ . Using  $F^{-1} = F^*$  and  $M_{\Gamma}^* = M_{\Gamma}^{-1} = M_{\Gamma}$  it follows that  $F^{-1} = \tilde{F} M_{\Gamma}$ , and equation (A.9) is an immediate consequence.  $\square$

The following result shows that every “spectral set” is a “spectrum” and conversely that every “spectrum” is a “spectral set”.

**Corollary A.3.** *Let  $\mu$  be a positive Borel measure on a locally compact group  $G$  and let  $\nu$  be a positive Borel measure on the dual group  $\Gamma = \hat{G}$ . Then the following are equivalent:*

- (i)  $(\mu, \nu)$  is a spectral pair on  $(G, \Gamma)$ ,
- (ii)  $(\tilde{\nu}, \mu)$  is a spectral pair on  $(\Gamma, G)$ ,
- (iii)  $(\tilde{\mu}, \tilde{\nu})$  is a spectral pair on  $(G, \Gamma)$ ,
- (iv)  $(\nu, \tilde{\mu})$  is a spectral pair on  $(\Gamma, G)$ ,

where  $\tilde{\mu}(\Delta) = \mu(-\Delta)$  and  $\tilde{\nu}(\Delta') = \nu(-\Delta')$ , for any Borel sets  $\Delta$  in  $G$  and  $\Delta'$  in  $\Gamma$ .

Corollary A.3 generalizes the well-known case where  $G = \Gamma = \mathbb{R}^d$ ,  $\mu = \nu = m$  and  $F$  is the usual Fourier transform. For example (iii) corresponds to the fact that  $(F^2 f)(\xi) = f(-\xi)$  for the usual Fourier transform.

**Theorem A.4.** *If  $(\mu, \nu)$  is a spectral pair, then so is  $(\nu, \mu)$ .*

*Proof.* Suppose  $(\mu, \nu)$  is a spectral pair, then  $(\nu, \tilde{\mu})$  is a spectral pair by A.3, hence

$$F_1 g(x) := \langle e_x, g \rangle_\nu = \int g(\lambda) \overline{e_x(\lambda)} d\nu(\lambda)$$

determines an isometric isomorphism mapping  $L^2(\nu)$  onto  $L^2(\tilde{\mu})$ . We must show that

$$F_2 g(x) := \langle \overline{e_x}, g \rangle_\nu = \int g(\lambda) e_x(\lambda) d\nu(\lambda)$$

determines an isometric isomorphism mapping  $L^2(\nu)$  onto  $L^2(\mu)$ . But this is easy since

$$\begin{aligned} F_2 g(x) &= \langle \overline{e_x}, g \rangle_\nu \\ &= \langle e_{-x}, g \rangle_\nu \\ &= F_1 g(-x) \\ &= M_G F_1 g(x) \end{aligned}$$

so  $F_2 = M_G F_1$ . By construction of  $\tilde{\mu}$  from  $\mu$  it follows that  $M_G$  is an isometric isomorphism mapping  $L^2(\tilde{\mu})$  onto  $L^2(\mu)$ , hence  $F_2 = M_G F_1$ , being the composition of two isometric isomorphisms, is an isometric isomorphism as we needed to show.  $\square$

If  $(\mu, \nu)$  is a spectral pair, then we may define a strongly continuous unitary representation  $U$  of  $G$  on  $\mathcal{L}^2(\mu)$  by

$$(A.10) \quad (F(U(t)f))(\xi) := e_t(\xi)(Ff)(\xi)$$

for  $f \in \mathcal{L}^2(\mu)$ ,  $t \in G$  and  $\nu$ -a.e.  $\xi \in \Gamma$ .

We have the following generalization of [Ped87, Corollary 1.11], it shows that if  $(\mu, \nu)$  is a spectral pair of measures and  $\mu$  is a restriction of Haar measure to a set of finite measure, then the pair  $(\mu, \nu)$  corresponds to a spectral pair of sets.

**Theorem A.5.** *Let  $(\mu, \nu)$  be a spectral pair. The following are equivalent:*

- (i)  $\mu(G) < \infty$ ;
- (ii)  $\nu$  is a constant multiple of a counting measure;
- (iii)  $\nu(\{\xi\}) \neq 0$  for some  $\xi \in \Gamma$ .

*In the affirmative case, the constant in (ii) is  $\mu(G)^{-1}$ .*

*Proof.* Since (ii)  $\Rightarrow$  (iii) is trivial, we will show that (i)  $\Rightarrow$  (ii) and (iii)  $\Rightarrow$  (i).

*Proof of (i)  $\Rightarrow$  (ii).* If  $\mu(G) < \infty$ , then  $e_\xi \in \mathcal{L}^2(\mu)$  for any  $\xi \in \Gamma$ . It follows that

$$(A.11) \quad \langle U(t)e_\xi | e_\eta \rangle_\mu = \langle FU(t)e_\xi | Fe_\eta \rangle_\mu = \overline{e_t(\xi)} \langle Fe_\xi | Fe_\eta \rangle_\mu = \overline{e_t(\xi)} \langle e_\xi | e_\eta \rangle_\mu$$

for  $\nu$ -a.e.  $\xi, \eta \in \Gamma$ . Now  $U(t)^* = U(-t)$  so  $\langle U(t)e_\xi | e_\eta \rangle_\mu = \langle e_\xi | U(-t)e_\eta \rangle_\mu$  and therefore  $\overline{e_t(\xi)} \langle e_\xi | e_\eta \rangle_\mu = e_{-t}(\eta) \langle e_\xi | e_\eta \rangle_\mu$  for any  $t \in G$  and  $\nu$ -a.e.  $\xi, \eta \in \Gamma$ . So, either  $\xi = \eta$ , or  $\langle e_\xi | e_\eta \rangle_\mu = 0$ . Consequently,

$$(A.12) \quad (Fe_\xi)(\eta) = \int e_\xi(x) \overline{e_\eta(x)} d\mu(x) = \langle e_\eta | e_\xi \rangle_\mu = \begin{cases} 0 & \text{if } \eta \neq \xi \\ \mu(G) & \text{if } \eta = \xi. \end{cases}$$

It follows that

$$(A.13) \quad \mu(G) = \langle e_\xi | e_\xi \rangle_\mu = \langle Fe_\xi | Fe_\xi \rangle_\nu = \int |Fe_\xi(\eta)|^2 d\nu(\eta) = \mu(G)^2 \nu(\{\xi\}).$$

Hence,  $\nu(\{\xi\}) = \mu(G)^{-1}$ , for any  $\xi \in \text{supp } \nu$ .

*Proof of (iii)  $\Rightarrow$  (i).* By Theorem A.4 it is sufficient to show that if  $\mu(\{x\}) \neq 0$  for some  $x \in G$  then  $\nu(\Gamma) < \infty$ . Suppose first that  $x \in G$  is such that  $\mu(\{x\}) \neq 1$ . Since we can rescale  $\mu$  and  $\nu$  by the same constant we may assume  $\mu(\{x\}) = 1$ . Let

$$(A.14) \quad \delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

Then

$$(A.15) \quad (F\delta_x)(\xi) = \int \delta_x(y) \overline{e_\xi(y)} d\mu(y) = \overline{e_\xi(x)} = e_{m_G(x)}(\xi),$$

so  $e_{m_G(x)} \in \mathcal{L}^2(\nu)$  and

$$(A.16) \quad 1 = \|\delta_x\|_\mu^2 = \|e_{m_G(x)}\|_\nu^2 = \nu(\Gamma).$$

An application of Theorem A.1 completes the proof.  $\square$

**Corollary A.6.** *If  $(\mu, \nu)$  is a spectral pair such that  $\mu(G) < \infty$ , then  $\{e_\xi : \xi \in \text{supp } \nu\}$  is an orthogonal basis for  $\mathcal{L}^2(\mu)$ .*

*Proof.* This is a simple consequence of the proof of Theorem A.5.  $\square$

Our next goal is to show that  $\mu$  has uniform density. We first show that  $U(t)$  acts by translation under appropriate circumstances.

**Lemma A.7.** *Suppose  $(\mu, \nu)$  is a spectral pair. Let  $\mathcal{O} \subset G$  be  $\mu$ -measurable and let  $t \in G$ . If  $\mathcal{O} \cup (\mathcal{O} + t) \subset \text{supp } \mu$ , then*

$$(A.17) \quad (U(t)f)(x) = f(x+t)$$

for  $\mu$ -a.e.  $x \in \mathcal{O}$  and every  $f \in \mathcal{L}^2(\mu)$ .

*Proof.* If  $Ff \in \mathcal{L}^1 \cap \mathcal{L}^2(\nu)$ , then

$$(A.18) \quad \begin{aligned} (U(t)f)(x) &= F^{-1}(e_t(\xi)(Ff)(\xi))(x) \\ &= \int e_t(\xi) e_x(\xi) (Ff)(\xi) d\nu(\xi) \\ &= (F^{-1}Ff)(x+t) \\ &= f(x+t) \end{aligned}$$

for  $\mu$ -a.e.  $x \in \mathcal{O}$ .  $\square$

**Corollary A.8.** *If  $(\mu, \nu)$  is a spectral pair,  $t \in G$ , and  $\mathcal{O} \subset G$  is  $\mu$ -measurable, then the inclusion  $\mathcal{O} \cup (\mathcal{O} + t) \subset \text{supp } \mu$  implies that*

$$(A.19) \quad \mu(\mathcal{O}) = \mu(\mathcal{O} + t).$$

*Proof.* If  $x \in \mathcal{O} + t$ , then

$$(A.20) \quad (U(-t)\chi_{\mathcal{O}})(x) = \chi_{\mathcal{O}}(x-t) = 1,$$

hence

$$(A.21) \quad \mu(\mathcal{O}) = \|\chi_{\mathcal{O}}\|_{\mu}^2 = \|U(-t)\chi_{\mathcal{O}}\|_{\mu}^2 \leq \mu(\mathcal{O} + t).$$

Similarly, if  $x \in \mathcal{O}$ , then

$$(A.22) \quad (U(t)\chi_{\mathcal{O}+t})(x) = \chi_{\mathcal{O}+t}(x+t) = 1,$$

so

$$(A.23) \quad \mu(\mathcal{O} + t) = \|\chi_{\mathcal{O}+t}\|_{\mu}^2 = \|U(t)\chi_{\mathcal{O}+t}\|_{\mu}^2 \leq \mu(\mathcal{O}).$$

The desired equality is immediate.  $\square$

Our Corollary A.8 is related to the discussion in the recent paper [KoLa96] of “tiling the line by translates of a function” as follows: By Corollary A.8 such tilings do not come from spectral sets.

**Proposition A.9.** *Let  $(\mu, \nu)$  be a spectral pair. Each  $\xi \in \Gamma$  is contained in an open set  $\mathcal{O}_{\xi}$  with  $\nu(\mathcal{O}_{\xi}) < \infty$ . In particular, each compact subset of  $\Gamma$  has finite  $\nu$ -measure; in short,  $\nu$  is a Radon measure.*

*Proof.* Let

$$(A.24) \quad \mathcal{N} := \{\xi \in \Gamma : (Ff)(\xi) = 0 \text{ for all } f \in \mathcal{L}^1 \cap \mathcal{L}^2(\mu)\}.$$

By density of  $\mathcal{L}^1 \cap \mathcal{L}^2(\mu)$  in  $\mathcal{L}^2(\mu)$ , we have  $\nu(\mathcal{N}) = 0$ . If  $f \in \mathcal{L}^1 \cap \mathcal{L}^2(\mu)$  then  $Ff$  is continuous and

$$\begin{aligned} (Ff)(\xi + \eta) &= \int f(x) \overline{e_{\xi+\eta}(x)} d\mu(x) \\ &= \int (\bar{e}_{\eta}f)(x) \overline{e_{\xi}(x)} d\mu(x) \\ &= (F(\bar{e}_{\eta}f))(\xi), \end{aligned}$$

so  $\xi \in \mathcal{N} \Rightarrow \xi + \eta \in \mathcal{N}$ , hence  $\mathcal{N} = \emptyset$ . Using  $\mathcal{N} = \emptyset$ , we see that there exist  $f_{\xi} \in \mathcal{L}^1 \cap \mathcal{L}^2(\mu)$  such that  $(Ff_{\xi})(\eta) \geq 1$  for  $\eta$  in some neighborhood  $\mathcal{O}_{\xi}$  of  $\xi$  and  $\nu(\mathcal{O}_{\xi}) \leq \|Ff_{\xi}\|_{\nu}^2 = \|f_{\xi}\|_{\mu}^2$ , as desired.  $\square$

**Proposition A.10.** *Let  $(\mu, \nu)$  be a spectral pair. If  $\mu(G) < \infty$ , then  $\text{supp } \nu$  is uniformly discrete in the sense that there exists an open set  $\mathcal{O} \subset \Gamma$  such that  $0 \in \mathcal{O}$  and*

$$(A.25) \quad (\mathcal{O} + \xi) \cap \text{supp } \nu = \{\xi\}$$

for all  $\xi \in \text{supp } \nu$ .

*Proof.* Let  $g(\eta) := \langle e_0 | e_{\eta} \rangle_{\mu}$ . Then  $g$  is continuous and  $g(0) = \mu(G)^{\frac{1}{2}}$ . Let  $\mathcal{O} \subset \Gamma$  be an open set such that  $0 \in \mathcal{O}$  and  $|g(\eta)| > \frac{1}{2}\mu(G)^{\frac{1}{2}}$  for any  $\eta \in \mathcal{O}$ . Then

$$(A.26) \quad |\langle e_{\xi} | e_{\eta} \rangle| = |g(\eta - \xi)| > \frac{1}{2}\mu(G)^{\frac{1}{2}}$$

whenever  $\eta \in \mathcal{O} + \xi$ .  $\square$

**Theorem A.11** (Uncertainty Principle). *Let  $(\mu, \nu)$  be a spectral pair and  $f \in L^2(\mu)$  with  $\|f\|_\mu > 0$ . If  $A, B$  are measurable sets in  $G$  and  $\Gamma$ , respectively, such that*

$$\begin{aligned} \text{(i)} \quad & \|f - \chi_A f\|_\mu \leq \varepsilon \|f\|_\mu, \\ \text{(ii)} \quad & \|Ff - \chi_B Ff\|_\nu \leq \delta \|Ff\|_\nu = \delta \|f\|_\mu \end{aligned}$$

*both hold, then*

$$\text{(iii)} \quad (1 - \varepsilon - \delta)^2 \leq \mu(A) \nu(B).$$

*Proof (sketch).* If  $f \in \mathcal{L}^2(\mu)$ ,  $A \subset G$  and  $B \subset \Gamma$ , then

(A.27)

$$\begin{aligned} \|\chi_A F^{-1} \chi_B Ff\|_\mu^2 &= \int_G \chi_A(x) \left| \int_\Gamma \chi_B(\xi) e_x(\xi) \int_G f(y) \overline{e_\xi(y)} d\mu(y) d\nu(\xi) \right|^2 d\mu(x) \\ &= \int_G \chi_A(x) \left| \int f(y) \int \chi_B(\xi) e_{x-y}(\xi) d\nu(\xi) d\mu(y) \right|^2 d\mu(x) \\ &= \int_G \chi_A(x) \left| \int f(y) \overline{(F^{-1} \chi_B \bar{e}_x)(y)} d\mu(y) \right|^2 d\mu(x) \\ &\leq \int_G \chi_A(x) \|f\|_\mu^2 \|F^{-1} \chi_B \bar{e}_x\|_\mu^2 d\mu(x) \\ &= \|f\|_\mu^2 \int_G \chi_A(x) \|\chi_B \bar{e}_x\|_\nu^2 d\mu(x) \\ &= \|f\|_\mu^2 \mu(A) \nu(B). \end{aligned}$$

If furthermore  $\|f - \chi_A f\|_\mu \leq \varepsilon \|f\|_\mu$  and  $\|Ff - \chi_B Ff\|_\nu \leq \delta \|Ff\|_\nu$ , then also

$$\begin{aligned} \text{(A.28)} \quad \|f\|_\mu - \|\chi_A F^{-1} \chi_B Ff\|_\mu &\leq \|f - \chi_A F^{-1} \chi_B Ff\|_\mu \\ &\leq \|f - \chi_A f\|_\mu + \|\chi_A f - \chi_A F^{-1} \chi_B Ff\|_\mu \\ &\leq \varepsilon \|f\|_\mu + \|f - F^{-1} \chi_B Ff\|_\mu \\ &= \varepsilon \|f\|_\mu + \|Ff - \chi_B Ff\|_\nu \\ &\leq \varepsilon \|f\|_\mu + \delta \|f\|_\mu, \end{aligned}$$

so

$$(1 - \varepsilon - \delta) \|f\|_\mu \leq \|\chi_A F^{-1} \chi_B Ff\|_\mu,$$

completing the proof.  $\square$

There is a vast literature on Uncertainty Principles. Theorem A.11 and its proof are modelled after [DoSt89], see also [deJe94], [Be85]. A comprehensive recent survey is [FoSi97]. A much more detailed analysis of operators similar to  $\chi_A F^{-1} \chi_B F$  appears in [Lan67].

**Corollary A.12.** *Let  $(\mu, \nu)$  be a spectral pair. If there exists a sequence of  $\mu$ -measurable sets  $A_n \subset G$  such that  $\mu(A_n) \neq 0$ , and  $\mu(A_n) \rightarrow 0$ , then  $\nu(\Gamma) = +\infty$ .*

*Proof.* Let  $A \subset G$  be measurable with  $0 < \mu(A) < \infty$  and let  $f = \mu(A)^{-\frac{1}{2}} \chi_A$ . Then  $\|f\|_\mu = 1$  and

$$\text{(A.29)} \quad 1 \leq \mu(A) \nu(\Gamma)$$



by the Uncertainty Principle with  $B = \Gamma$ . The desired conclusion is immediate.  $\square$

Corollary A.12 should be compared to Theorem A.5. If  $(\mu, \nu)$  is a spectral pair and  $\mu$  is Lebesgue measure restricted to a set  $\Omega \subset \mathbb{R}^d$  of finite nonzero Lebesgue measure, then Theorem A.5 and Corollary A.12 imply that  $\nu$  is  $\mu(\Omega)^{-1}$  times counting measure on an infinite set  $\Lambda \subset \mathbb{R}^d$ . Finally, if  $G$  is a finite abelian group,  $\Gamma$  is the dual group,  $\mu$  is counting measure on  $G$ , and  $\nu$  is  $\mu(G)^{-1}$  times counting measure on  $\Gamma$ , then  $(\mu, \nu)$  is a spectral pair; see [HeRo63]. In particular, the assumption in Corollary A.12 that there exist sets  $A \subset G$  of arbitrarily small  $\mu$ -measure cannot be removed.

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